

INFINITE WEIGHTED GRAPHS WITH BOUNDED RESISTANCE METRIC

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To the memory of Ola Bratteli

ABSTRACT. We consider infinite weighted graphs G , i.e., sets of vertices V , and edges E assumed countable infinite. An assignment of weights is a positive symmetric function c on E (the edge-set), conductance. From this, one naturally defines a reversible Markov process, and a corresponding Laplace operator acting on functions on V , voltage distributions. The harmonic functions are of special importance. We establish explicit boundary representations for the harmonic functions on G of finite energy.

We compute a resistance metric d from a given conductance function. (The resistance distance $d(x, y)$ between two vertices x and y is the voltage drop from x to y , which is induced by the given assignment of resistors when 1 amp is inserted at the vertex x , and then extracted again at y .)

We study the class of models where this resistance metric is bounded. We show that then the finite-energy functions form an algebra of $\frac{1}{2}$ -Lipschitz-continuous and bounded functions on V , relative to the metric d . We further show that, in this case, the metric completion M of (V, d) is automatically compact, and that the vertex-set V is open in M . We obtain a Poisson boundary-representation for the harmonic functions of finite energy, and an interpolation formula for every function on V of finite energy.

We further compare M to other compactifications; e.g., to certain path-space models.

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1. INTRODUCTION

We consider a certain class of infinite weighted graphs G . They are specified by prescribed sets of vertices V , and edges E ; countable infinite. An assignment of weights, is a positive symmetric function c of E (the edge-set). In electrical network models, the function c represents conductance, and its reciprocal resistance. So fixing a conductance function is then equivalent to an assignment of resistors on the edges of G . From this, one naturally defines a reversible Markov process, and a corresponding Laplace operator (called graph Laplacian) acting on functions on V , the vertex-set. Functions on V typically represent voltage distributions, and the harmonic functions are of special importance. For list of explicit details required on (V, E, c) , we refer to the details in Section 2.

We will be especially interested in boundary representations for harmonic functions of finite energy.

From a given conductance function, we compute a resistance metric d . Intuitively, the resistance distance $d(x, y)$ between two vertices x and y is the voltage drop from x to y , which is induced by the given assignment of resistors when 1 amp is inserted at the vertex x , and then extracted again at y . We study the realistic class of models when this resistance metric is assumed bounded. In this case the finite-energy functions form an algebra of continuous and bounded functions on V , relative to the metric d . We further show that, in this case, the metric completion M of (V, d) is automatically compact. The vertex-set V is open in M , and we obtain a Poisson boundary-representation for the harmonic functions of finite energy.

A number of additional properties are established for M . In particular, we compare M to other compactifications in the literature; e.g., to path-space models.

There is a recent increased interest in analysis on large (infinite) networks, motivated by a host of applications; see e.g., [JP10, JP11, AJV14, KPS12, AK12]. We shall be citing standard facts from the general theory. In addition, we use facts from analysis, Hilbert space geometry, potential theory, boundaries, and Markov measures; see e.g., [TB13, CXY15, Sko13, Her12, DJ11, Rob11, BKY14].

2. BASIC SETTINGS

Let $G = (V, E, c)$ be a weighted graph, where $c =$ conductance function (see Definition 2.1), $V =$ vertex-set (countable infinite), and the edges $E \subset V \times V \setminus \{\text{diagonal}\}$ such that:

- (G1) $(x, y) \in E \iff (y, x) \in E; x, y \in V;$
- (G2) $0 < \#\{y \in V \mid (x, y) \in E\} < \infty$, for all $x \in V;$
- (G3) Connectedness: $\exists o \in V$ s.t. for all $y \in V \exists x_0, x_1, \dots, x_n \in V$ with $x_0 = o$, $x_n = y$, $(x_{i-1}, x_i) \in E, \forall i = 1, \dots, n$.
- (G4) If a conductance function c is given, we require $c_{x_{i-1}x_i} > 0$.

Definition 2.1. A function $c : E \rightarrow \mathbb{R}_+ \cup \{0\}$ is called *conductance function* if

- (1) $c(e) \geq 0, \forall e \in E;$
- (2) Given $x \in V, c_{xy} > 0, c_{xy} = c_{yx}$, for all $(xy) \in E;$
- (3) If $(x, y) \in E$, we write $x \sim y$; and it is assumed that $\#\{y \in V \mid y \sim x\}$ is finite for all $x \in V$.

If $x \in V$, we set

$$c(x) := \sum_{x \sim y} c_{xy}, \quad \text{where } x \sim y \stackrel{\text{Def}}{\iff} (xy) \in E. \quad (2.1)$$

Let $G = (V, E, c)$ be as above. Assume G is connected, i.e., there is a base point o in V such that every $x \in V$ is connected to o via a finite path of edges, see (G3).

Set $\mathcal{H}_E :=$ completion of functions $u : V \rightarrow \mathbb{C}$ with respect to

$$\langle u, v \rangle_{\mathcal{H}_E} := \frac{1}{2} \sum_{(x,y) \in E} \sum_{(x,y) \in E} c_{xy} (\overline{u(x)} - \overline{u(y)}) (v(x) - v(y)) \quad (2.2)$$

$$\|u\|_{\mathcal{H}_E}^2 := \frac{1}{2} \sum_{(x,y) \in E} \sum_{(x,y) \in E} c_{xy} |u(x) - u(y)|^2 \quad (2.3)$$

(or simply all functions u s.t. the sum in (2.3) is finite.) Then \mathcal{H}_E is a Hilbert space [JP10]. (\mathcal{H}_E is known to be bigger than the \mathcal{H}_E -norm completion of the finitely supported functions on V . We know that the non-constant harmonic functions on V are *not* in the \mathcal{H}_E -completion of the finitely supported functions, see Remark 7.1.)

Lemma 2.2 ([JP10]). (i) For every pair of vertices $x, y \in V$, there is a unique $v_{x,y} \in \mathcal{H}_E$ (unique up to an additive constant) such that

$$f(x) - f(y) = \langle v_{xy}, f \rangle_{\mathcal{H}_E}, \quad \forall f \in \mathcal{H}_E. \quad (2.4)$$

(ii) The vector v_{xy} in (2.4) satisfies

$$\Delta v_{xy} = \delta_x - \delta_y, \quad (2.5)$$

where $(\Delta f)(u) := \sum_{y \sim u} c_{uy} (f(u) - f(y))$.

Remark 2.3. The solution to (2.5) is not unique: If v_{xy} satisfies (2.5), and if $h \in \mathcal{H}_E$ satisfies $\Delta h = 0$ (harmonic), then $v_{xy} + h$ also satisfies (2.5); but generally *not* (2.4).

Let $V' := V \setminus \{o\}$, and set

$$v_x := v_{x,o}, \quad \forall x \in V'. \quad (2.6)$$

Corollary 2.4. For all $x, y \in V$, $\exists!$ real-valued dipole vector $v_{xy} \in \mathcal{H}_E$ s.t.

$$\langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y), \quad \forall u \in \mathcal{H}_E. \quad (2.7)$$

Moreover,

$$v_{xy} - v_{zy} = v_{xz}, \quad \forall x, y, z \in V. \quad (2.8)$$

Definition 2.5. Fix a weighted graph (connected), set the *graph Laplacian* $\Delta = \Delta_c$, where

$$(\Delta u)(x) = \sum_{y \sim x} c_{xy} (u(x) - u(y)) = c(x) u(x) - \sum_{y \sim x} c_{xy} u(y) \quad (2.9)$$

for all functions u on V .

Lemma 2.7 below summarizes the key properties of Δ as an operator, both in $l^2(V)$ and in \mathcal{H}_E .

Definition 2.6. Let (V, E, c) and Δ be as outlined, and let \mathcal{H}_E be the corresponding energy-Hilbert space; see (2.3). Finally let $l^2 = l^2(V)$ denote the usual l^2 -space, i.e., all $w : V \rightarrow \mathbb{C}$ such that

$$\|w\|_{l^2}^2 = \sum_{x \in V} |w(x)|^2 < \infty.$$

We shall need the subspace $\mathcal{D}_2 \subset l^2$ (dense in the l^2 -norm):

$$\mathcal{D}_2 := \text{span} \{ \delta_x \mid x \in V \}. \quad (2.10)$$

If $\{v_x \mid x \in V'\}$ denotes a system of dipoles (see (2.6)), we set $\mathcal{D}_E \subset \mathcal{H}_E$ (dense in \mathcal{H}_E -norm):

$$\mathcal{D}_E := \text{span} \{ v_x \mid x \in V' \}; \quad (2.11)$$

in both cases “span” means all finite linear combinations.

We show in Section 8 that $l^2(V)$ contains *no* non-constant harmonic functions; but \mathcal{H}_E generally does.

Lemma 2.7. *The following hold:*

- (1) $\langle \Delta u, v \rangle_{l^2} = \langle u, \Delta v \rangle_{l^2}, \forall u, v \in \mathcal{D}_2;$
- (2) $\langle \Delta u, v \rangle_{\mathcal{H}_E} = \langle u, \Delta v \rangle_{\mathcal{H}_E}, \forall u, v \in \mathcal{D}_E;$
- (3) $\langle u, \Delta u \rangle_{l^2} \geq 0, \forall u \in \mathcal{D}_2$, and
- (4) $\langle u, \Delta u \rangle_{\mathcal{H}_E} \geq 0, \forall u \in \mathcal{D}_E.$

As a densely defined operator in $l^2(V)$, Δ is essentially selfadjoint; but, as an operator with dense domain in \mathcal{H}_E , Δ is generally not essentially selfadjoint.

Moreover, we have

- (5) $\langle \delta_x, u \rangle_{\mathcal{H}_E} = (\Delta u)(x), \forall x \in V, \forall u \in \mathcal{H}_E.$
- (6) $\Delta v_{xy} = \delta_x - \delta_y, \forall x, y \in V$, where $v_{xy} \in \mathcal{H}_E$. In particular, $\Delta v_x = \delta_x - \delta_o$, $x \in V' = V \setminus \{o\}.$
- (7)

$$\delta_x(\cdot) = c(x) v_x(\cdot) - \sum_{y \sim x} c_{xy} v_y(\cdot), \forall x \in V'.$$

(8)

$$\langle \delta_x, \delta_y \rangle_{\mathcal{H}_E} = \begin{cases} c(x) = \sum_{t \sim x} c_{xt} & \text{if } y = x \\ -c_{xy} & \text{if } (xy) \in E \\ 0 & \text{if } (xy) \notin E, x \neq y \end{cases}$$

Proof. See [JP10, JP11, Jor08]. For the selfadjointness of the graph Laplacian in $l^2(V)$, see Theorem 2.9 below. \square

Remark 2.8. We will show in Section 9 that the two $\infty \times \infty$ matrices

$$\Delta_{xy} := \langle \delta_x, \delta_y \rangle_{\mathcal{H}_E}, \text{ (see (8)); and} \quad (2.12)$$

$$K_{xy} := \langle v_x, v_y \rangle_{\mathcal{H}_E} \quad (2.13)$$

are formal inverses; more precisely, for any $x, y \in V$, the following $\infty \times \infty$ matrix-product, ΔK and $K\Delta$ are well defined; and

$$\sum_{z \in V'} \Delta_{xz} K_{zy} = \delta_{x,y}, \text{ and} \quad (2.14)$$

$$\sum_{z \in V'} K_{xz} \Delta_{zy} = \delta_{x,y} \quad (2.15)$$

both hold. However, the operator theoretic interpretation of the two, (2.14) vs (2.15), is different.

Theorem 2.9 ([Jor08, JP10, JP11, Woj07, KL12]). *Let $G = (E, V, c)$ be a weighted graph as specified above; so with a given conductance function c defined on the set of edges E of G ; and let Δ be the corresponding Laplace operator. Then, as an operator in $l^2(V)$ with domain consisting of finitely supported functions, Δ is essentially selfadjoint.*

Proof. Below we give a new proof of this essential selfadjointness. One advantage with the proof below is its use of different properties of the operator Δ than was the case for earlier approaches. We also believe that the idea used here has wider use; – that it is applicable to other operators in analysis and potential theory, both discrete and continuous.

Note TFAE:

- (i) $f \in l^2(V)$ is a Δ -defect vector;
- (ii) $\langle \varphi + \Delta\varphi, f \rangle_{l^2} = 0, \forall \varphi \in \text{span}\{\delta_x\}$;
- (iii) $\langle \delta_x(\cdot) + (\Delta\delta_x)(\cdot), f \rangle_{l^2} = 0, \forall x \in V$;
- (iv) $\langle \underbrace{\delta_x + c(x)\delta_x}_{(1+c(x))\delta_x} - \sum_{y \sim x} c_{xy}\delta_y, f \rangle_{l^2} = 0, \forall x \in V$;
- (v) $(1 + c(x))f(x) - \sum_{y \sim x} c_{xy}f(y) = 0, \forall x \in V$;
- (vi) $(1 + c(x))f(x) - c(x)(\mathbb{P}f)(x) = 0, \forall x \in V$; where

$$p_{xy} = \frac{1}{c(x)}c_{xy} \quad \text{and} \quad (\mathbb{P}f)(x) = \sum_{y \sim x} p_{xy}f(y);$$

- (vii) $(\mathbb{P}f)(x) = \left(1 + \frac{1}{c(x)}\right)f(x), \forall x \in V$.

With the splitting $f = \Re\{f\} + i\Im\{f\}$, it is enough to consider the case when f is real valued.

Since $f \in l^2(V)$, it has a local max, i.e., $\exists x_0 \in V$ s.t. $f(\cdot) \leq f(x_0)$ in V . Assume $f(x_0) > 0$ (otherwise replace f by $-f$.) Now, if f is a defect vector, we have

$$\left(1 + \frac{1}{c(x_0)}\right) f(x_0) \underset{(\text{by } (vi))}{=} (\mathbb{P}f)(x_0) \leq f(x_0) \implies \frac{1}{c(x_0)} f(x_0) \leq 0,$$

which contradicts the assumption that $f(x_0) > 0$.

□

Theorem 2.10. *Let $(V, E, c, \Delta, \mathcal{H}_E)$ be as above, and fix a base-point $o \in V$. Set $V' := V \setminus \{o\}$. Fix dipole $v_x := v_{x,o}$, $x \in V'$ s.t.*

$$f(x) - f(o) = \langle v_x, f \rangle_{\mathcal{H}_E}, \quad \forall f \in \mathcal{H}_E, \forall x \in V'. \quad (2.16)$$

Set

$$(\Delta^{-1})_{xy} := \langle v_x, v_y \rangle_{\mathcal{H}_E}, \quad (x, y) \in V' \times V'. \quad (2.17)$$

Then Δ is not essentially selfadjoint on $\mathcal{D}_E := \text{span}\{v_x \mid x \in V'\}$ if and only if there is a non-zero function $f \in \mathcal{H}_E$ such that

$$h(x) := f(x) + \sum_{y \in V'} (\Delta^{-1})_{xy} f(y) \quad (2.18)$$

is harmonic.

Proof. By general operator theory (see [DS88]), the essential selfadjointness assertion holds iff the following implication holds:

$$\left[f \in \mathcal{H}_E, \text{ and } \langle \varphi + \Delta\varphi, f \rangle_{\mathcal{H}_E} = 0, \forall \varphi \in \mathcal{D}_E \right] \implies [f = 0]. \quad (2.19)$$

Taking $\varphi = v_x$, and modulo an additive constant, we see that a possible solution $f \in \mathcal{H}_E$ to (2.19) will have the form, setting:

$$(\mathbb{P}f)(x) = \left(1 + \frac{1}{c(x)}\right) f(x), \quad \forall x \in V', \quad (2.20)$$

where $(\mathbb{P}f)(x) = \sum_{y \sim x} p_{x,y} f(y)$, $p_{x,y} = \frac{c_{x,y}}{c(x)}$.

An iteration of (2.20) yields

$$(\mathbb{P}^{n+1}f)(x) = f(x) + \sum_{k=0}^n \mathbb{P}^k \left(\frac{f}{c} \right)(x). \quad (2.21)$$

But we have pointwise convergence on the RHS in (2.21), and

$$(1 - \mathbb{P})^{-1} = \left(\frac{1}{c} \Delta \right)^{-1}, \text{ so}$$

$$\begin{aligned}
(1 - \mathbb{P})^{-1} \left(\frac{f}{c} \right) (x) &= \Delta^{-1} (\text{diag}(c)) \left(\frac{f}{c} \right) (x) \\
&= (\Delta^{-1} f) (x) = \sum_y (\Delta^{-1})_{xy} f(y). \tag{2.22}
\end{aligned}$$

So the LHS in (2.21) must converge pointwise; but it is clear that $h = \lim_n \mathbb{P}^n f$ is harmonic.

Finally, it is clear that every solution $f \in \mathcal{H}_E$ to (2.18) will satisfy (2.19); which in turn is the equation which decides non-essential selfadjointness, by general theory. \square

Remark 2.11. We introduce the Markov measure $\mu^{(Markov)}$ on the space Ω of all $G = (V, E)$ -paths, and the Markov-walk process

$$\pi_n(\omega) := \omega_n, \quad \forall \omega \in \Omega, \quad n \in \mathbb{N}_0, \tag{2.23}$$

where $\omega = (\omega_0, \omega_1, \omega_2, \dots)$, $\omega_j \in V$, $(\omega_j, \omega_{j+1}) \in E$, $\forall j \in \mathbb{N}_0$. Then the matrix product \mathbb{P}^k in (2.21) is

$$\text{Prob}(\{\pi_{m+k} = y \mid \pi_m = x\}) = (\mathbb{P}^k)_{x,y}. \tag{2.24}$$

We shall return to this Markov-process in Section 8 below.

3. FROM CONDUCTANCE TO CURRENT FLOW

Let $G = (V, E, c)$ be an infinite weighted graph (connected, see (G4) before Definition 2.1). Here, V = vertex-set, E = edges, and $c : E \rightarrow \mathbb{R}_+$ is a fixed conductance function, so that $c = (c_{xy})$, $(xy) \in E$. Let \mathcal{H}_E be the corresponding energy-Hilbert space (see (2.2)-(2.3)).

Set the current flow $I_{(xy)} := \partial w$, where

$$I_{(xy)} = (\partial w)(x, y) = c_{xy}(w(x) - w(y)), \quad \forall (xy) \in E, \quad w \in \mathcal{H}_E. \tag{3.1}$$

And set

$$Dissp = \left\{ \partial w \mid w \in \mathcal{H}_E, \quad \|\partial w\|_{Diss}^2 := \sum \sum \frac{1}{c_{xy}} I_{(xy)}^2 < \infty \right\} \tag{3.2}$$

as a weighted l^2 -space on E , where $1/c_{xy}$ = resistance.

On edges $(u, v) \in E$ from x to y in V , the current $I_{(u,v)}$ is

$$I_{uv} = c_{uv}(f(u) - f(v))$$

where f denotes a voltage-distribution. See Fig 3.1.

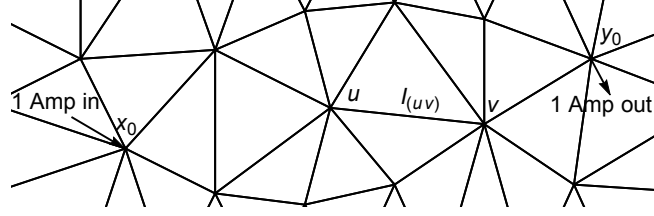


FIGURE 3.1. Current flows from vertex x_0 to vertex y_0 ; with a given conductance function c . The convex set $W_{x_0 y_0}$.

Lemma 3.1. *The operator $\partial : \mathcal{H}_E \rightarrow \text{Dissp}$ is isometric.*

Proof. One checks that

$$\|w\|_{\mathcal{H}_E}^2 = \sum \sum c_{xy} |w(x) - w(y)|^2 \quad (\text{energy}) \quad (3.3)$$

$$= \sum \sum \frac{1}{c_{xy}} I_{(xy)}^2 \quad (\text{dissipation}) \quad (3.4)$$

where $I_{xy} = (\partial w)_{xy} = c_{xy}(w(x) - w(y))$, and $1/c_{xy}$ = resistance on the edge (xy) , see (3.1); and the lemma follows. \square

Definition 3.2. Set $\text{dist}(x_0, y_0)$ = distance $x_0 \rightarrow y_0$ = voltage drop from x_0 to y_0 when current I satisfies $I = 1$ at x_0 “in” and current $I = -1$ at y_0 “out.”

Theorem 3.3. *There is a unique current flow s.t.*

$$\text{dist}(x_0, y_0) = \inf \left\{ \|I\|_{\text{Diss}}^2, I|_{\{x_0, y_0\}} = 1 \text{ Amp in, and } 1 \text{ out} \right\} \quad (3.5)$$

Proof. Recall that by Lemma 2.4, $\exists! v_{xy}$ s.t.

$$\langle v_{xy}, f \rangle_{\mathcal{H}_E} = f(x) - f(y), \quad \forall (x, y) \in V \times V, \quad \forall f \in \mathcal{H}_E. \quad (3.6)$$

Set $I = \partial v_{xy}$, then

$$\begin{aligned} d(x_0, y_0) &= \inf \|I\|_{\text{Diss}}^2 = \|\partial v_{x_0 y_0}\|_{\text{Diss}}^2 \\ &= \|v_{x_0 y_0}\|_{\mathcal{H}_E}^2 (= \text{resistance distance}); \end{aligned} \quad (3.7)$$

i.e., the infimum in (3.5) is obtained at the flow $I = \partial v_{x_0 y_0}$, see (3.6)-(3.7). For a proof, see [JP10, JP11].

The infimum in (3.5) and (3.7) is justified with the following Hilbert space geometry applied to the energy-Hilbert space \mathcal{H}_E :

The infimum in (3.5) is attained when $I_0 = \partial v_{x_0 y_0}$. We use that I_0 is the vector in the convex set $W_{x_0 y_0}$ of minimum norm. Since ∂ from Lemma 3.1 is isometric,

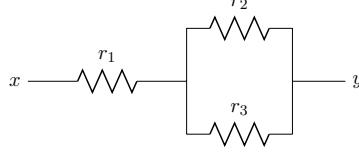


FIGURE 3.2. Example of resistor configuration in a network: configuration of three resistors, having values r_1, r_2, r_3 Ohm.

we see that $W_{x_0 y_0}$ is both closed and convex. From Hilbert space geometry, see e.g. [Rud91], we know that $W_{x_0 y_0}$ contains a vector of smallest norm. From the definition of $W_{x_0 y_0}$ (see e.g., Fig 3.1), we conclude that the minimum must be $I_0 = \partial v_{x_0 y_0}$; see also [JP11]. \square

Remark 3.4. The function $v_{x_0 y_0}$ in (3.7) is called a *dipole*, and it satisfies

$$\Delta v_{x_0 y_0} = \delta_{x_0} - \delta_{y_0} \quad (3.8)$$

where Δ is the Laplacian from (2.9).

Below, we offer seven different, but equivalent, formulas for the resistance metric $d_{res}(x, y)$:

Theorem 3.5 ([JP11]). *Let V, E, c, Δ , and d_{res} be as above; let $x, y \in V$, and let W_{xy} be the set from Fig 3.1. Then*

$$\begin{aligned} d_{res}(x, y) &= v(x) - v(y) \text{ when } v = v_{x,y} \\ &= \|v_{x,y}\|_{\mathcal{H}_E}^2 \\ &= \min \left\{ \|I\|_{Dissp}^2 : I \in W_{x,y} \right\} \\ &= \|w\|_{\mathcal{H}_E}^2 \text{ when } \Delta w = \delta_x - \delta_y \\ &= \frac{1}{\min \left\{ \|w\|_{\mathcal{H}_E}^2 : w \in \mathcal{H}_E, |w(x) - w(y)| = 1 \right\}} \\ &= \inf \left\{ K : |w(x) - w(y)|^2 \leq K \|w\|_{\mathcal{H}_E}^2, w \in \mathcal{H}_E \right\} \\ &= \sup \left\{ |w(x) - w(y)|^2 : w \in \mathcal{H}_E, \|w\|_{\mathcal{H}_E} \leq 1 \right\}. \end{aligned}$$

Example 3.6 (see Fig 3.2).

$$d_{res}(x, y) = r_1 + \frac{1}{\frac{1}{r_2} + \frac{1}{r_3}}$$

4. THE METRIC BOUNDARY

Definition 4.1. By M we mean the set of equivalence classes of sequences $(x_i) \subset V$ of vertices s.t.

$$\lim_{i,j \rightarrow \infty} d(x_i, x_j) = 0 \text{ (Cauchy); and} \quad (4.1)$$

$$(x_i) \sim (y_i) \stackrel{(\text{Def})}{\iff} \lim_{i \rightarrow \infty} d(x_i, y_i) = 0. \quad (4.2)$$

The vertex-set V is identified with a subset of M via the mapping $\gamma : V \rightarrow M$,

$$V \ni x \longmapsto \gamma(x) = \text{class}(x, x, x, \dots). \quad (4.3)$$

Hence $b \in M \setminus V (= \text{bdd } V)$ iff $b = (y_i) \in M$ satisfies the following:

$$\forall x \in V, \exists \varepsilon \in \mathbb{R}_+, \exists (y_{i_k}) \subset (y_i), \text{ s.t. } d(x, y_{i_k}) \geq \varepsilon, \quad \forall k \in \mathbb{N}. \quad (4.4)$$

Note that the assertion in (4.4) states that:

$$d(\gamma(x), b) > 0, \quad \forall x \in V. \quad (4.5)$$

We now show that if $d := d_{res}$ is *bounded*, then every function $f \in \mathcal{H}_E$ extends by closure to M : If $b \in M$, and $\{x_i\} \subset V$, are such that $\lim_{i \rightarrow \infty} d(x_i, b) = 0$, we set

$$\tilde{f}(b) = \lim_{i \rightarrow \infty} f(x_i). \quad (4.6)$$

It is then immediate that

$$|\tilde{f}(b) - \tilde{f}(b')|^2 \leq d(b, b') \|f\|_{\mathcal{H}_E}^2. \quad (4.7)$$

Theorem 4.2. *If the resistance metric $d = d_{res}$ is bounded on $V \times V$, then*

$$\mathcal{H}_E \subset l^\infty(V), \text{ and } \widetilde{\mathcal{H}_E} \subseteq C(M); \quad (4.8)$$

i.e., every energy function w on V is bounded, and \mathcal{H}_E is an algebra under pointwise product.

Proof. The containment in (4.8) follows from the estimate (5.6).

We proceed to show that \mathcal{H}_E is an algebra when (V, d) is assumed bounded:

Let $u, w \in \mathcal{H}_E$, then

$$(uw)(x) := u(x)w(x), \quad \forall x \in V;$$

satisfies

$$\|uw\|_{\mathcal{H}_E}^2 \leq \left(\|u\|_\infty^2 + \|w\|_\infty^2 \right) \left(\|u\|_{\mathcal{H}_E}^2 + \|w\|_{\mathcal{H}_E}^2 \right). \quad (4.9)$$

Since $u, w \in l^\infty(V)$, it follows that $uw \in \mathcal{H}_E$, i.e., $\|uw\|_{\mathcal{H}_E} < \infty$.

The proof of (4.9) is as follows:

$$\begin{aligned}
& \sum_E c_{xy} |(uw)(x) - (uw)(y)|^2 \\
&= \sum_E c_{xy} |u(x)(w(x) - w(y)) + w(y)(u(x) - u(y))|^2 \\
&\stackrel{(\text{Schwarz})}{\leq} \sum_E c_{xy} \left(|u(x)|^2 + |w(y)|^2 \right) \left(|u(x) - u(y)|^2 + |w(x) - w(y)|^2 \right) \\
&\leq \left(\|u\|_\infty^2 + \|w\|_\infty^2 \right) \left(\sum_E c_{xy} |u(x) - u(y)|^2 + \sum_E c_{xy} |w(x) - w(y)|^2 \right)
\end{aligned}$$

which is the desired estimate. \square

Remark 4.3. After the completion of a first version of our paper, D. Lenz kindly informed us that a version of Theorem 4.2 above is in the paper [GHK⁺14]. Our approach and aim here is different though.

Corollary 4.4. *Let $V, E, c, d = d_{res}$ be as above, i.e., assume that d is bounded, and that M is compact; then when the constant function $\mathbb{1}$ on M is adjoined*

$$\widetilde{\mathcal{H}}_E = \{\tilde{f} \mid f \in \mathcal{H}_E\} \subset C(M) \quad (4.10)$$

is a dense subalgebra; dense in the uniform norm on $C(M)$.

Proof. We already proved that $\widetilde{\mathcal{H}}_E$ is an algebra of continuous functions on M (= the metric completion of (V, d_{res})), so we only need to show that it is dense in the $\|\cdot\|_\infty$ -norm on M . Since M is compact, $\|\tilde{f}\|_\infty = \max\{|\tilde{f}(b)| : b \in M\}$.

It is clear that $\widetilde{\mathcal{H}}_E$ is closed under complex conjugation; so, by the Stone-Weierstrass theorem, we only need to prove that it separates points. We will prove that if $b \neq b'$ in M then there is a vertex $x \in V$ such that $\tilde{v}_x(b) \neq \tilde{v}_x(b')$.

Since M is the metric completion of (V, d) , it is enough to show that $\widetilde{\mathcal{H}}_E$ separate points in V . Assume the contrary: that there are vertices $y, z \in V$, $y \neq z$ such that $v_x(y) = v_x(z)$ holds for all $x \in V$; in other words, $\langle v_x, v_y - v_z \rangle_{\mathcal{H}_E} = 0$ holds for all $x \in V$. But $\text{span}\{v_x \mid x \in V\}$ is dense in \mathcal{H}_E ; and so $v_y - v_z = 0$, contradicting $d(y, z) = \|v_y - v_z\|_{\mathcal{H}_E}^2 > 0$. \square

5. DISCRETE RESISTANCE METRIC – METRIC COMPLETIONS

Set $d := d_{res}$ the resistance metric, see (3.7). Let (M, \tilde{d}) be the metric completion of (V, d) , i.e., V consists of a metric space M with the metric

$$d_{res}(x, y) = \|v_{xy}\|_{\mathcal{H}_E}^2 \quad (5.1)$$

where v_{xy} is the *dipole* vector in (2.7).

It is an important theorem [JP11, JP10] that d_{res} in (5.1) is indeed a metric, i.e., that

$$d_{res}(x, y) \leq d_{res}(x, z) + d_{res}(z, y) \quad (5.2)$$

holds for $\forall x, y, z \in V$. This result applies to all weighted graph models where d_{res} is computed from a fixed conductance function $c : E \rightarrow \mathbb{R}_+$.

Let

$$K(x, y) = \langle v_x, v_y \rangle_{\mathcal{H}_E}, \quad (5.3)$$

then the triangle inequality (5.2) is equivalent to

$$K(x, y) \geq K(x, z) + K(z, y) - K(z, z), \quad \forall x, y, z \in V. \quad (5.4)$$

Now assume that $d = d_{res}$ is *bounded*.

Definition 5.1. We say that a system (V, E, c, d_{res}) is type *A* if whenever $\lim_j v_{x_j}$ exists in $C(V, d)$ then (x_j) is a Cauchy sequence in (V, d) .

Theorem 5.2. *If d_{res} is bounded on $V \times V$, and assume the system (V, E, c, d_{res}) is of type A; then (M, \tilde{d}) is a compact metric space.*

Proof. Fix a base-point $o \in V$, and set $v_x = v_{x,o}$, $x \in V \setminus \{o\}$, then

$$v_{xy} = v_x - v_y \quad (5.5)$$

as follows from (5.4); also see Lemma 2.4. By Schwarz, applied to the energy Hilbert space $(\mathcal{H}_E, \langle \cdot, \cdot \rangle_{\mathcal{H}_E})$, we get the following Lipschitz-estimate:

$$|f(x) - f(y)|^2 \leq d(x, y) \|f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}_E, \quad x, y \in V. \quad (5.6)$$

Consequences of (5.4)-(5.6):

- (1) Every $f \in \mathcal{H}_E$ extends to a uniformly continuous function \tilde{f} on M ; extension by metric limits.

- (2) If $x_i \in V$, and $d(x_i, x_j) \rightarrow 0$, for $i, j \rightarrow \infty$, then $f(x_i)$ has a limit in \mathbb{C} (or \mathbb{R}).
Set

$$\tilde{x} \in M, \quad \tilde{x} = \lim_i x_i. \quad (5.7)$$

If $(x_i), (y_i) \subset V$ are Cauchy sequences, set (the extended metric \tilde{d}):

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{i \rightarrow \infty} d(x_i, y_i); \quad (5.8)$$

then by (5.6), we get

$$|\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{y})|^2 \leq \tilde{d}(\tilde{x}, \tilde{y}) \|f\|_{\mathcal{H}_E}^2. \quad (5.9)$$

The assertion in the theorem follows from the considerations below. \square

Lemma 5.3. *An application of Arzelà–Ascoli shows that*

$$\left\{ \tilde{f} \subset C(M) \mid f \in \mathcal{H}_E, \|f\|_{\mathcal{H}_E} \leq 1 \right\} \quad (5.10)$$

is relatively compact in $C(M)$, in the Montel topology of uniform convergence on compact sets.

But if d is bounded on $V \times V$, then

$$\|v_{x_i}\|_{\mathcal{H}_E} \leq A \cdot K_d \quad (5.11)$$

where A is a fixed global constant, since

$$d(x_i, x_j) = \|v_{x_i} - v_{x_j}\|_{\mathcal{H}_E}^2. \quad (5.12)$$

Hence by (5.10) with K_d in place of 1, we get that:

Corollary 5.4. *Assume type A, then for every sequence x_1, x_2, x_3, \dots in V , \exists subsequence (x_{i_k}) s.t.*

- (i) $\lim_{k \rightarrow \infty} x_{i_k} = b \in M$; and
- (ii) Let $f_{lim} \in C(M)$ be the limit of the subsequence $\{\tilde{v}_{x_i}\} \subset C(M)$, then

$$\lim_{k \rightarrow \infty} \tilde{v}_{x_{i_k}}(b) = f_{lim}(b).$$

Proof. To see that $b \in M$, note that

$$d(x_{i_k}, x_{i_l}) = \|v_{x_{i_k}} - v_{x_{i_l}}\|_{\mathcal{H}_E}^2 = |\tilde{v}_{x_{i_k}}(x_{i_k}) - \tilde{v}_{x_{i_l}}(x_{i_l})| \xrightarrow{k, l \rightarrow \infty} 0;$$

since by (5.9), the functions $\tilde{v}_{x_{i_k}}(\cdot)$ are uniformly bounded, and equicontinuous on M . Since we assume the system (V, E, c, d_{res}) is of type A, it follows that every sequence

x_1, x_2, \dots in V has a convergence subsequence with limit in M . By the definition of M , the same is true for M , and so M is compact: Every sequence $b_1, b_2, \dots \subset M$ contains a convergent subsequence. \square

Remark 5.5. The following example from [GHK⁺14] shows that our assumed condition “type A” in Theorem 5.2 and Corollary 5.4 cannot be omitted. There are bounded resistance metrics (non-type A) for which the corresponding completions are non-compact.

We learned from D. Lenz that the boundedness of the resistance metric does not imply the completion (M, \tilde{d}) is compact [GHK⁺14]. Indeed, the type A assumption for the system (V, E, c, d_{res}) is required. (See Definition 5.1.)

Example 5.6 (Example 8.6 in [GHK⁺14]). Fig 5.1 below is a tree-like graph with many ends all of which have bounded distance to the root (making the resistance metric bounded) but at the same time being too far apart from each other to be covered by finitely many balls of an fixed but arbitrarily small size. Thus, the weighted graph in this case is bounded with respect to d_{res} metric and the completion is not compact with respect to the resistance metric.

The graph basically consists of a copy of the natural numbers with the property that each natural number has a ray emanating from it and this ray being again the natural numbers. There are weights (Fig 5.2) on the graph making all these copies of the natural numbers having bounded diameter in the resistance metric. This makes the resistance metric on this graph bounded. On the other hand a point far out in one of the emanating rays has a uniform distance to any point far out in any other emanating ray. This makes the example non-totally bounded. Hence, the example has the mentioned properties.

Lemma 5.7. *Let $G = (V, E, c)$ be the weighted graph in Example 5.6. Fix a base-point $o \in V$, and set $\mathcal{D}_E = \text{span} \{v_x \mid x \in V \setminus \{o\}\}$ (see (2.11)). Then $\Delta|_{\mathcal{D}_E}$, as a densely defined Hermitian operator in the energy-Hilbert space \mathcal{H}_E , is not essentially selfadjoint. Moreover, the deficiency indices are (∞, ∞) .*

Proof. Let the c be the conductance function as specified in Fig 5.1-5.2. Recall that

$$\begin{aligned} (\Delta f)(x) &= \sum_{x \sim y} c_{xy} (f(x) - f(y)) = c(x) \left(f(x) - \sum_{x \sim y} p_{xy} f(y) \right) \\ &= c(I - \mathbb{P})f(x), \text{ for all functions } f \text{ on } V; \end{aligned}$$

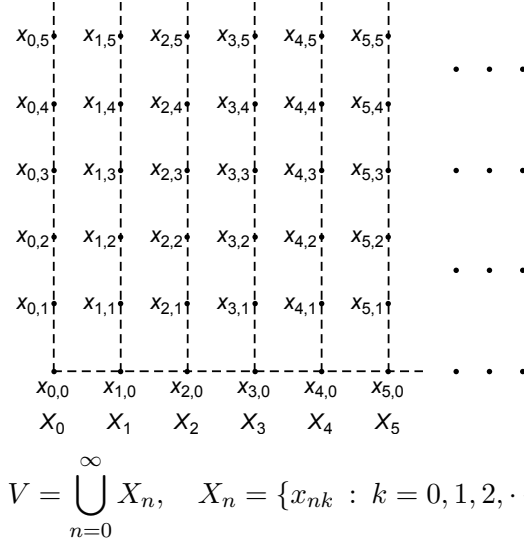


FIGURE 5.1. A double infinite planar graph: An infinitely long comb as an infinite array of teeth, each tooth infinitely long.

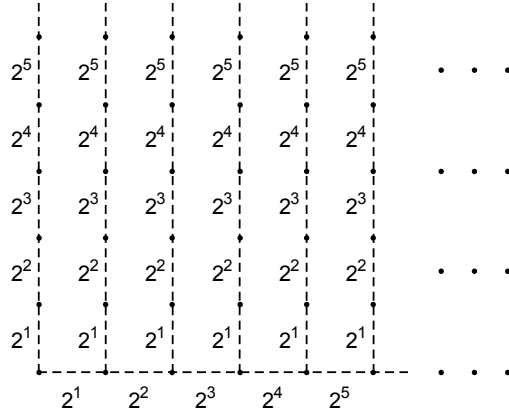


FIGURE 5.2. The conductance function $c_{(xy)}$.

where $c(x) = \sum_{y \sim x} c_{xy}$, $p_{xy} = c_{xy}/c(x) =$ transition probability, and $(\mathbb{P}f)(x) = \sum_{y \sim x} p_{xy}f(y)$. Also see Theorem 2.10, Example 7.6, 7.13, and Remark 7.9.

Suppose f is a defect vector for Δ . Since Δ is positive, it suffices to consider $\Delta f = -f$. Note that

$$\Delta f = -f \iff c(I - \mathbb{P})f = -f \iff \mathbb{P}f = \left(1 + \frac{1}{c}\right)f. \quad (5.13)$$

We proceed to show that f is in \mathcal{H}_E , i.e., $\|f\|_{\mathcal{H}_E} < \infty$.

Let $V = \{x_{n,k}\}$ be the vertex-set as specified in Fig 5.1. Then, we have

$$c(x_{n,k}) = 2^k + 2^{k+1} \quad (5.14)$$

$$p_{x_{n,k}, x_{n,k-1}} = \frac{2^k}{2^k + 2^{k+1}} = \frac{1}{3} \quad (5.15)$$

$$p_{x_{n,k}, x_{n,k+1}} = \frac{2^{k+1}}{2^k + 2^{k+1}} = \frac{2}{3} \quad (5.16)$$

and so

$$\begin{aligned} (\mathbb{P}f)(x_{n,k}) &= \frac{1}{3}f(x_{n,k-1}) + \frac{2}{3}f(x_{n,k+1}), \text{ and} \\ \left(1 + \frac{1}{c}\right)f(x_{n,k}) &= \left(1 + \frac{1}{2^k \cdot 3}\right)f(x_{n,k}); \text{ see (5.14)-(5.16).} \end{aligned}$$

Thus, the defect vector f satisfies $\Delta f = -f \iff$

$$\frac{1}{3}f(x_{n,k-1}) + \frac{2}{3}f(x_{n,k+1}) = \left(1 + \frac{1}{2^k \cdot 3}\right)f(x_{n,k}). \quad (5.17)$$

Set

$$l_k := l_{n,k} = f(x_{n,k}); \quad (5.18)$$

then we get the following recursive equation:

$$\frac{1}{3}l_{k-1} + \frac{2}{3}l_{k+1} = \left(1 + \frac{1}{2^k \cdot 3}\right)l_k; \quad (5.19)$$

i.e.,

$$l_{k+1} = \frac{3}{2} \left[\left(1 + \frac{1}{2^k \cdot 3}\right)l_k - \frac{1}{3}l_{k-1} \right] = \left(\frac{3}{2} + \frac{1}{2^{k+1}}\right)l_k - \frac{1}{2}l_{k-1}.$$

Or, using matrix notation, we have

$$\begin{pmatrix} l_{k+1} \\ l_k \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + \frac{1}{2^{k+1}} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} l_k \\ l_{k-1} \end{pmatrix}. \quad (5.20)$$

The asymptotic estimate of the sequence (l_k) can be derived from the eigenvalues of the coefficient matrix in (5.20). Note the eigenvalues are given by

$$x_{\pm} = \frac{\frac{3}{2} - \frac{1}{2^{k+1}} \pm \sqrt{\left(\frac{3}{2} - \frac{1}{2^{k+1}}\right)^2 - 2}}{2} \sim \frac{\frac{3}{2} \pm \frac{1}{2}}{2}, \quad \text{asymptotically.}$$

Conclusion. The root $x_- = \frac{1}{2}$ shows that $l_k \sim 1/2^k$ so $f(x_{n,k}) \sim 1/2^k$ asymptotically. Consequently,

$$\begin{aligned} \|f\|_{\mathcal{H}_E}^2 &\sim \sum_k 2^k \left(\frac{1}{2^k} - \frac{1}{2^{k+1}} \right)^2 + \sum_n 2^n \left(\frac{1}{2^n} - \frac{1}{2^{n+1}} \right)^2 \\ &\sim \sum_k \frac{1}{2^k} + \sum_n \frac{1}{2^n} < \infty. \end{aligned}$$

Therefore, the corresponding defect vector f is in \mathcal{H}_E , and so $\Delta|_{\mathcal{H}_E}$ is not essentially selfadjoint. \square

Set the Gelfand space $G_E = \{\beta : \mathcal{H}_E \rightarrow \mathbb{C}(\text{or } \mathbb{R})\}$ (see [Rud91]) s.t.

$$\beta(uw) = \beta(u)\beta(w), \quad \forall u, w \in \mathcal{H}_E; \quad (5.21)$$

i.e., *multiplicative functionals*.

Definition 5.8. Let $M :=$ metric completion of (V, d_{res}) . Set

$$(x_i) \sim (y_i) \stackrel{\text{Def}}{\iff} d_{res}(x_i, y_i) \rightarrow 0$$

for all Cauchy sequences $(x_i), (y_i) \subset V$.

Theorem 5.9. $M \subset G_E$, see (5.21). (The metric completion is contained in the Gelfand space.)

Proof. Every $w \in \mathcal{H}_E$ extends by closure to M , by

$$\tilde{w}(\tilde{x}) = \lim_{i \rightarrow \infty} w(x_i), \quad \text{if } d_{res}(x_i, x_j) \rightarrow 0. \quad (5.22)$$

To see this, use the estimate $|w(x) - w(y)|^2 \leq d(x, y) \|w\|_{\mathcal{H}_E}^2$, $\forall w \in \mathcal{H}_E$; see (5.6).

Now, set $\beta_{\tilde{x}}(w) = \tilde{w}(\tilde{x})$, and note that (5.21) is then immediate. (In fact, M is a compact metric space if d_{res} is bounded.) \square

Remark 5.10. It was proved in [GHK⁺14] that the Gelfand space is the Royden compactification; see [GHK⁺14] for details.

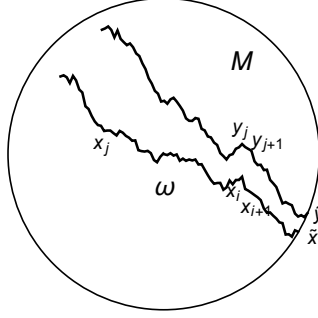


FIGURE 5.3. $\psi(\omega) = \tilde{x}$; Ω = infinite path-space; M = metric completion of (V, d_{res}) .

Question 5.11. *In these examples, what is the connection between (1) M , (2) G_E , and (3) the ∞ path space Ω models? (Recall the $\omega \in \Omega$, when $\omega = (x_i)_{i \in \mathbb{N}}$, where $x_i \in V$, $(x_i x_{i+1}) \in E$.)*

Remark 5.12. In some cases $(x_i)_{i \in \mathbb{Z}}$, $(x_i x_{i+1}) \in E$ s.t.

$$d_{res}(x_i, x_j) \xrightarrow{i, j \rightarrow \infty} 0, \quad (5.23)$$

there is a mapping $\psi : \Omega \rightarrow M \rightarrow G_E$. Note that (5.23) holds if d_{res} is bounded. In this case \exists continuous extension $\Omega \xrightarrow{\psi} M$, with $\psi(\omega) = \tilde{x}$ where $\omega = (x_i)_{i \in \mathbb{Z}} \in \Omega$, and where $\tilde{x} \in M$ is the metric limit $\tilde{d}_{res}(x_i, \tilde{x}) \rightarrow 0$ if $d(x_i, x_j) \rightarrow 0$. See Fig 5.3.

Actually even if not every path $\omega = (x_i)_{i \in \mathbb{Z}}$ satisfies $d_{res}(x_i, x_j) \rightarrow 0$, we can pass to a sub-sequence.

Theorem 5.13. *Assume that d_{res} is type A and bounded on $V \times V$ (thus (M, \tilde{d}_{res}) is compact by Theorem 5.2), and that $\omega = (x_i)_{i \in \mathbb{Z}} \in \Omega$, then \exists subsequence $\{x_{i_1}, x_{i_2}, \dots\} \subset \omega$, and $\exists \tilde{x} \in M$ s.t.*

$$\tilde{d}_{res}(x_{i_k}, \tilde{x}) \xrightarrow{k \rightarrow \infty} 0.$$

Proof. (Application of Arzelà–Ascoli) Recall that $v_i := v_{x_i, o}$, where

$$|v_i(z)|^2 = |\langle v_i, v_z \rangle|^2 \leq d(i, o) d(z, o) \leq K;$$

which implies that

$$|v_i(z) - v_i(z')|^2 \leq K d(z, z').$$

By Arzelà–Ascoli, \exists a subsequence s.t. $v_{i_k} - v_{i_l} \rightarrow 0$ in \mathcal{H}_E , as $d(x_{i_k}, x_{i_l}) \xrightarrow{k, l \rightarrow \infty} 0$. \square

6. POISSON-REPRESENTATIONS

Let $G = (V, E)$ be as above, and let $c : E \rightarrow \mathbb{R}_+$ be a fixed conductance function. Let $d = d_{res}$ be the corresponding resistance metric.

Our standard assumptions on G, c are as outlined in Section 2 above.

We assume in addition that

- (1) $\#V = \aleph_0$, i.e., countable infinite.
- (2) d_{res} is bounded on $V \times V$.
- (3) For all $x \in V$, $\exists \varepsilon = \varepsilon_x$ s.t.

$$\{y \in V \mid d(x, y) < \varepsilon_x\} = \{x\}, \text{ the singleton.} \quad (6.1)$$

We shall denote by M the metric completion of (V, d_{res}) , and identify V as a subset of M in the usual way, where

$$x \in V \longleftrightarrow \text{class}(x, x, x, \dots) \in M \quad (\infty \text{ repetition of vertex } x) \quad (6.2)$$

Proposition 6.1. *For $n \in \mathbb{N}$, set $w = (z_1, \dots, z_n)$ where $z_i \in V$ (vertices), a finite word, and denote by $(w\underline{x})$ the concatenation sequence*

$$(z_1, z_2, \dots, z_n, \underbrace{x, x, x, x, x, x, \dots}_{\text{infinite repetition}}); \quad (6.3)$$

we set $\underline{x} = (x, x, x, x, \dots)$; then $\gamma(x) = \{\underline{x}\} \cup \{w\underline{x}\}$, as w ranges over all finite words.

Proof. If $(y_i)_{i=1}^\infty$ is a sequence of vertices s.t. $\lim_{i \rightarrow \infty} d(y_i, x) = 0$, then, since x is isolated by (3), see (6.1), there must be a $n \in \{0, 1, 2, \dots\}$ such that $y_i = x$ for all $i \geq n$; and the desired conclusion follows. \square

Theorem 6.2. *Let $G = (V, E)$, c , d_{res} satisfying the conditions above, including (1)-(3) (so d_{res} is bounded). Then*

$$B := M \setminus V \quad (6.4)$$

is closed in M ; and for every $x \in V$, there is a Borel probability measure μ_x on B , i.e., $\mu_x \in M_1(B)$ such that, for all harmonic functions h on V with $\|h\|_{\mathcal{H}_E} < \infty$, we have

$$h(x) = \int_B \tilde{h}(b) d\mu_x(b) \quad (6.5)$$

where \tilde{h} is the extension $\in C(M)$ of h , obtained by metric completion, and where the function on the RHS in (6.5) is $\tilde{h}|_B$.

Proof. By Corollary 4.4, every $f \in \mathcal{H}_E$ has a unique continuous extension \tilde{f} to M ; and

$$\left| \tilde{f}(b) - \tilde{f}(b') \right|^2 \leq d(b, b') \|f\|_{\mathcal{H}_E}^2 \quad (6.6)$$

holds for $\forall b, b' \in M$. By (3), (Section 5), V identifies as an open subset in M , and so $B = M \setminus V$ is closed; and therefore compact. Recall M is compact by Theorem 5.2.

Recall from Section 2, that a function h on V is harmonic iff $\mathbb{P}h = h$, where

$$(\mathbb{P}h)(x) = \sum_{y \sim x} p_{xy} h(y) \quad (6.7)$$

and $p_{xy} := c_{xy}/c(x)$, for $\forall (xy) \in E$. Also recall, $(\Delta f)(x) = \sum_{y \sim x} c_{xy} (f(x) - f(y))$.

Hence the harmonic functions h in $\mathcal{H}_E \subset C(M)$ satisfy

$$\sup_{x \in V} |h(x)| = \left\| \tilde{h}|_B \right\|_{\infty}. \quad (6.8)$$

This is an application of (6.7) and a simple maxmin-principle.

Now set $\mathcal{A} \subset C(B)$ as follows:

$$\mathcal{A} = \left\{ \tilde{h}|_B; \mathbb{P}h = h, h \in \mathcal{H}_E \right\} \quad (6.9)$$

where “ $|_B$ ” denotes restriction; then, for every $x \in V$, the point-evaluation mapping:

$$\mathcal{A} \ni \tilde{h}|_B \longmapsto h(x) \quad (6.10)$$

defines a positive linear functional. Since $\mathbb{P}(\mathbb{1}) = \mathbb{1}$ where $\mathbb{1}$ is the constant one function, it follows that $\mathbb{1} \in \mathcal{A}$, and that $\mathbb{1} \mapsto 1$ in (6.10) (i.e., the functional in (6.10) attains value 1 on the constant function “one.”)

By the extension theorem of Banach and Krein, there is a positive linear functional on all of $C(B)$ which extends (6.10) from \mathcal{A} . By Riesz’ theorem, it is given by a unique probability measure $\mu_x \in M_1(B)$. Restricting to \mathcal{A} , and using (6.8), we get the desired formula (6.5); i.e., μ_x is the Poisson-kernel, and B is a Poisson-boundary, i.e., it reproduces the harmonic functions in \mathcal{H}_E . \square

7. CONTINUOUS VS DISCRETE: EXAMPLES

Remark 7.1. The orthogonal splitting

$$\mathcal{H}_E = \text{Harm} \oplus \overline{\text{span}}^{\mathcal{H}_E\text{-norm}} \{ \delta_x \mid x \in V \} \quad (7.1)$$

is often called the *Royden-decomposition* (see e.g., [Shi83, KM67]). There is a continuous analogy:

Let $\Omega \subset \mathbb{R}^d$ be bounded, and set

$$\begin{aligned} H^1(\Omega) &= \{f \text{ on } \Omega \mid \nabla f \in L^2(\Omega)\} / \text{constants} \\ \|f\|_{H^1(\Omega)}^2 &= \int_{\Omega} |\nabla f|^2 dx, \end{aligned} \tag{7.2}$$

and set

$$Harm_{\Omega} = \{f \in H^1(\Omega) \mid \Delta f = 0\}. \tag{7.3}$$

Then

$$H^1(\Omega) = Harm_{\Omega} \oplus \overline{C_c^{\infty}(\Omega)}^{H^1(\Omega)\text{-closure}};$$

i.e., we have the implication

$$\boxed{f \in H^1(\Omega), \text{ and } \langle f, \varphi \rangle_{H^1(\Omega)} = 0, \forall \varphi \in C_c^{\infty}(\Omega)} \implies \Delta f = 0.$$

7.1. Continuous models

Lemma 7.2. *Let*

$$\mathcal{H}_E = \{f : \mathbb{R} \rightarrow \mathbb{C} \mid \text{measurable}, f \in L^2, f' \in L^2\} \tag{7.4}$$

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \left(\int_{\mathbb{R}} |f|^2 + \int_{\mathbb{R}} |f'|^2 \right); \tag{7.5}$$

where f' in (7.4) denotes the weak-derivative of f .

(i) Then \mathcal{H}_E is a reproducing kernel Hilbert space (RKHS) consisting of bounded continuous functions.

(ii) Moreover, \mathcal{H}_E is an algebra under pointwise product with

$$\|fg\|_{\mathcal{H}_E} \leq \text{Const} \|f\|_{\mathcal{H}_E} \|g\|_{\mathcal{H}_E}, \quad \forall f, g \in \mathcal{H}_E.$$

Proof. \mathcal{H}_E is a RKHS with kernel

$$K(x, y) = e^{-|x-y|}. \tag{7.6}$$

To see this, set

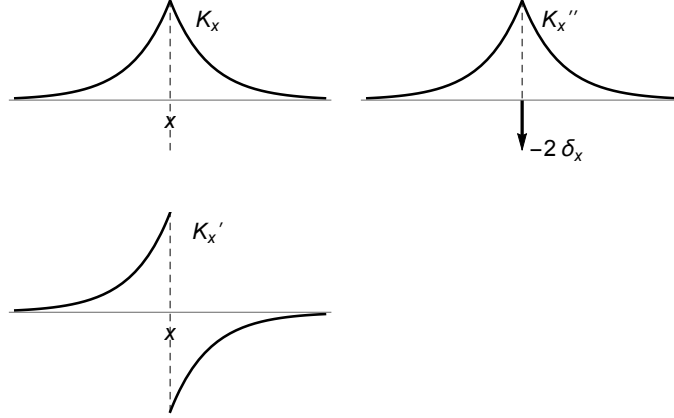
$$K_x(\cdot) = e^{-|x-\cdot|}, \tag{7.7}$$

and one checks that

$$\langle K_x, f \rangle_{\mathcal{H}_E} = \frac{1}{2} \left(\int_{\mathbb{R}} K_x f + \int_{\mathbb{R}} K'_x f' \right) = f(x), \quad \forall f \in \mathcal{H}_E, \forall x \in \mathbb{R}.$$

For a proof of part (ii), see [Jør81].

Note that $K''_x = K_x - 2\delta_x$ in the sense of distribution. See Fig 7.1. □

FIGURE 7.1. $K''_x = K_x - 2\delta_x$

Lemma 7.3. *If $f \in \mathcal{H}_E$, then f is bounded, and $\lim_{|x| \rightarrow \infty} f(x) = 0$.*

Proof. This follows from Riemann-Lebesgue, since

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi, \text{ and } \hat{f} \in L^1(\mathbb{R})$$

with $\sup_{x \in \mathbb{R}} |f(x)| \leq \|f\|_{\mathcal{H}_E}$, for all $f \in \mathcal{H}_E$. □

The resistance distance in this case is

$$\begin{aligned} d(x, y) &= \|K_x - K_y\|_{\mathcal{H}_E}^2 = 2 \left(1 - e^{-|x-y|}\right), \text{ and} \\ \sup_{x, y \in \mathbb{R}} d(x, y) &\leq 2. \end{aligned} \tag{7.8}$$

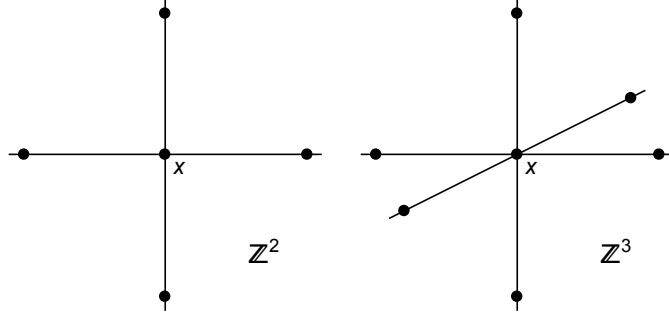
Hence the resistance metric d in (7.8) is bounded on \mathbb{R} , and the completion of \mathbb{R} with respect to d is the one-point compactification of \mathbb{R} , but for *discrete models*:

7.2. Discrete Models

Let $G = (V, E, c)$ be a weighted graph, with vertex-set V , edges E , and a fixed conductance function c . Let $d = d_{res}$ be the resistance metric, and we study the metric completion of G .

For functions on the \mathbb{Z} -lattice $L_d := \mathbb{Z}^d$, $d \geq 1$, fixed, set

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{x \sim y} e^{|x-y|} |f(x) - f(y)|^2; \tag{7.9}$$

FIGURE 7.2. Nearest neighbors for the lattices \mathbb{Z}^d , for $d = 2$ and $d = 3$.

where $x \sim y$, $x \neq y$, in (7.9) denotes *nearest neighbors*; and $x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$. (See Fig 7.2 for the case of $d = 2$ and $d = 3$.) Let

$$\mathcal{H}_E = \left\{ f \text{ on } \mathbb{Z}^d \mid \|f\|_{\mathcal{H}_E} < \infty \right\} \quad (7.10)$$

we set

$$|x - y| := \left(\sum_{i=1}^d |x_i - y_i|^2 \right)^{\frac{1}{2}}, \quad \forall x, y \in \mathbb{Z}^d. \quad (7.11)$$

Lemma 7.4. For $\forall x \in \mathbb{Z}^d$, we have the following: $\exists K = K_x < \infty$ s.t.

$$|f(x) - f(y)|^2 \leq K_x \|f\|_{\mathcal{H}_E}^2 \quad (\text{see Theorem 3.5.}) \quad (7.12)$$

Proof. Let $x \sim y$ denote nearest neighbors. Then $\exists! i$ s.t. $|x_i - y_i| = 1$, so $x_j - y_j = 0$ for $j \neq i$. The proof of (7.12) is standard. \square

Set

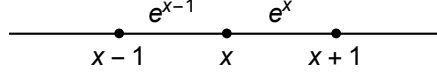
$$(\Delta f)(x) = \sum_{y \sim x} e^{|x-y|} (f(x) - f(y)), \quad \forall f \in \mathcal{H}_E.$$

Example 7.5. For $d = 1$, consider \mathbb{Z}_+ (see Fig 7.3), and

$$p_+ = \frac{e}{1+e}, \quad p_- = \frac{1}{1+e}.$$

A function u on \mathbb{Z}_+ is harmonic iff $I_x := e^x (u_{x+1} - u_x)$ is constant; and

$$\|u\|_{\mathcal{H}_E}^2 = \sum_x e^x (u_{x+1} - u_x)^2 = I_1^2 \sum_x e^{-x} = \frac{I_1^2}{e-1} < \infty.$$

FIGURE 7.3. \mathbb{Z}_+ conductance function c , $c_{x,x+1} = e^x$, $x \in \mathbb{Z}_+$.

Fix $0 < x < y$, then $v_{xy} = v_{yo}(t) - v_{xo}(t)$, where

$$v_{yo}(t) = \begin{cases} \sum_{i \leq y} e^{-i} & \text{if } t \leq y \\ \sum_{i=1}^y e^{-i} & \text{if } t > y \end{cases}$$

$$v_{xy}(t) = \begin{cases} 0 & \text{if } 0 < t \leq x \\ \sum_{i=x+1}^t e^{-i} & \text{if } x < t \leq y \\ \sum_{i=x+1}^y e^{-i} & \text{if } y \leq t, \quad t \in \mathbb{Z}_+ \end{cases}$$

and

$$d_{res}(x, y) = \sum_{i=x+1}^y e^{-i} = \frac{e^{-x} - e^{-y}}{e - 1};$$

and so d_{res} is clearly bounded.

But in this case the metric compactification is just the one-point compactification:

$$d_{res}(x, \infty) = \frac{e^{-x}}{e - 1}; \quad x \in \mathbb{Z}_+.$$

It follows, in these examples, that $B = M \setminus V$ is a singleton; so M is the one-point compactification.

Example 7.6. $V = \mathbb{Z}_+^d$, $d > 1$. Set $x + \varepsilon_i := (x_1, \dots, x_i + 1, x_{i+1}, \dots, x_d)$, for all $x \in \mathbb{Z}^d$. In this case we have:

$$(\Delta f)(x) = \sum_{i=1}^d e^{|x+\varepsilon_i|} (f(x) - f(x+\varepsilon_i)); \text{ and}$$

$$\|f\|_{\mathcal{H}_E}^2 = \sum_{x,i} e^{|x+\varepsilon_i|} |f(x) - f(x+\varepsilon_i)|^2$$

$$\sim \sum_{x \in \mathbb{Z}_+^d} e^{|x|} \sum_{i=1}^d |f(x+\varepsilon_i) - f(x)|^2 \longrightarrow 0$$

and $\|f\|_{\mathcal{H}_E}^2 < \infty \implies$

$$\exists N \quad \text{s.t.} \quad e^{|x|} \sum_i |f(x+\varepsilon_i) - f(x)|^2 < 1, \quad \forall x \in \mathbb{Z}_+^d, \forall |x| > N.$$

See Fig 7.4.

Hence,

$$\sum_i |f(x + \varepsilon_i) - f(x)|^2 < e^{-|x|}, \text{ asymptotic as } |x| \rightarrow \infty;$$

so that

$$f(x) \sim \text{Const} \cdot (1 - e^{-|x|}), \text{ asymptotic as } |x| \rightarrow \infty \text{ in } \mathbb{Z}_+^d.$$

Let $x \in L_d = \mathbb{Z}_+^d$, and

$$V_x \sim e^{-|x-t|}, \quad \forall t \in L_d$$

so that

$$\langle v_x, f \rangle = f(x) - f(o), \quad \forall x \in L_d, \forall f \in \mathcal{H}_E.$$

If $d \geq 3$, one checks that

$$\langle v_x, v_y \rangle_{\mathcal{H}_E} \simeq \frac{1}{|x - y|^{d-2}};$$

and as a result, is a one-point compactification, i.e., $B = M \setminus L_d = \{\infty\}$ the point at “infinity.”

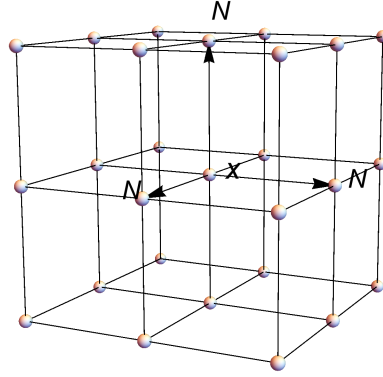


FIGURE 7.4. $d = 3$

Example 7.7. Let V = the binary tree, see Fig 7.5. If a vertex x in the tree is at level n , set

$$c_{(x,x+)} = c_+(n), \quad c_{(x,x-)} = c_-(n).$$

Then the arguments from above show that if $\sum_{n=1}^{\infty} \frac{1}{c_{\pm}(n)} < \infty$, then $B := M \setminus V$ is a Cantor-space.

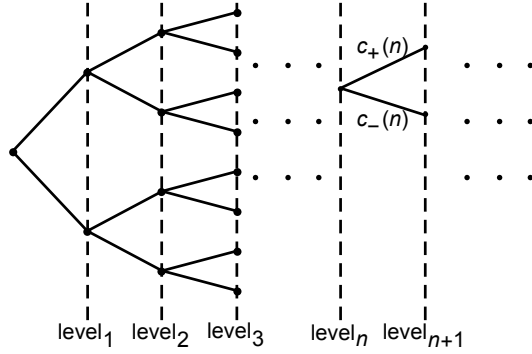


FIGURE 7.5. Binary tree with conductance

7.3. Bratteli diagrams

In our present papers, we considered weighted graphs $G = (V, E, c)$, vertices, edges and a weight (conductance) function. A Bratteli diagram is a special case of this, but the weighting usually doesn't refer to a conductance, but rather some kind of counting. In detail, if G is a Bratteli diagram, then its vertex set is stratified, by finite subsets V_n , called levels. While V is infinite, the sets V_n are finite. Then the requirement on G to be a Bratteli diagram is that the edges (lines in E) connect vertices from V_n to those at different levels; the nearest neighbor vertices are from level $n - 1$, and level $n + 1$. In its initial form (see [Bra72]) the Bratteli diagrams (later terminology) served as classification labels for approximately finite-dimensional C^* -algebras (also called AF-algebras, more precisely inductive limits of matrix-algebras). The need for such classification was initially motivated by physics. Subsequently, and initiated by George Elliott), the Bratteli diagrams acquired the structure of ordered groups (called K -groups), and the classification problem eventually took a rather complete form. But in the spirit of the original use of the diagrams from [Bra72], they have found many other uses in representation theory; the fundamental idea being that the lines (edges) are effective in classifying complicated systems of inclusions, i.e., counting the respective multiplicities in these inclusions of algebras, or representations, by numbers assigned to the edges. In this incarnation, they are even known as useful tools in the design of fast (finite) Fourier transforms.

And there are yet other applications; some deal with symbolic dynamics; see the papers in the bibliography, for example [HPS92], and measures on infinite path spaces obtained from “infinite strings of edges” from the given Bratteli diagram.

The papers [BJKR00] and [BJO04] deal with yet a different classification; that of order-isomorphism of the diagrams themselves. It turns out that the latter classification problem, in the general case, is so “complicated” that it has been proved to be undecidable. So in the Bratteli-Jorgensen et al. papers regarding this, we narrowed our focus to that of stationary Bratteli-diagrams; and we proved that then a classification is possible; even by explicit algorithms, and by explicit lists of numerical parameters.

Nonetheless the questions we consider here fall in a different category, and they don’t restrict the focus to stationary diagrams; even apply to graphs G which are not Bratteli diagrams.

If $\Delta = C - E$ as an $\infty \times \infty$ matrix representation, where

$$C = \text{diag}(c(x))_{x \in V} = \begin{pmatrix} c(x) & & & 0 \\ & \ddots & & \\ & & \text{diagonal} & \\ 0 & & & \end{pmatrix}$$

and

$$E = \begin{pmatrix} 0 & \ddots & & 0 \\ \ddots & 0 & c_{xy} & \\ & c_{xy} & \ddots & \ddots \\ 0 & & & \ddots \end{pmatrix}$$

symmetric, $c_{xy} > 0$; then

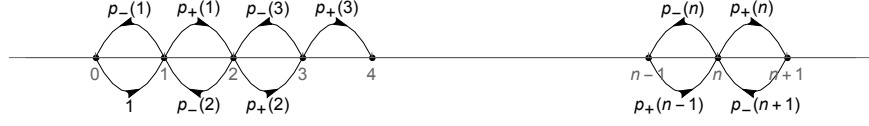
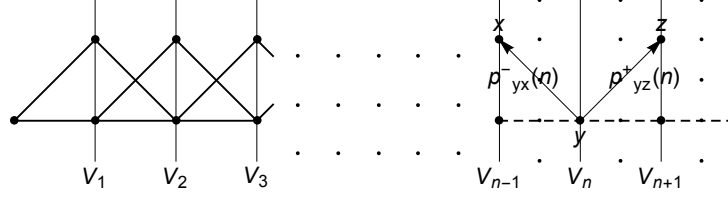
$$\Delta = (\Delta_{xy}) = C - E \tag{7.13}$$

where

$$\Delta_{xy} = \begin{cases} c(x) & \text{if } x = y \\ -c_{xy} & \text{if } (xy) \in E \text{ (recall } x \sim y \implies x \neq y) \\ 0 & \text{otherwise;} \end{cases}$$

and we get the Green’s function K as follows:

$$K = \langle v_x, v_y \rangle_{\mathcal{H}_E} \tag{7.14}$$

FIGURE 7.6. Transition probabilities $p_{\pm}(n)$, $n = 0, 1, 2, \dots$ FIGURE 7.7. A Bratteli diagram, formula (7.18) with vertex-set $V = \{\emptyset\} \cup V_1 \cup V_2 \cup \dots$ and transition between neighboring levels.

the Green's function of Δ satisfies

$$\sum_z \Delta_{xz} K_{zy} = \delta_{xy}, \quad (7.15)$$

and

$$\begin{aligned} \Delta^{-1} &= (C - E)^{-1} = (I - C^{-1}E)^{-1} C^{-1} \\ &= \sum_{n=0}^{\infty} (C^{-1}E)^n C^{-1} = G_P C^{-1}, \end{aligned} \quad (7.16)$$

where G_P is the Green's function of a Markov transition (see Fig 7.6). Note that C^{-1} is easy, since it is diagonal:

$$C = \text{diag}((c(x))_{x \in V}), \text{ then } C^{-1} = \text{diag}\left(\left(\frac{1}{c(x)}\right)_{x \in V}\right). \quad (7.17)$$

An example is (see Fig 7.6-7.7)

$$c(n) = c_n + c_{n+1}, \quad c_n > 0. \quad (7.18)$$

Lemma 7.8. *If (V, E, c) is constructed from a Bratteli diagram with levels V_1, V_2, \dots , then the Green's function K for Δ satisfies*

$$K = G_P C^{-1} \quad (7.19)$$

where G_P is the random-walk Green's function associated with a \pm Markov random walk, see (7.17) and Fig 7.6.

For Bratteli diagrams, see e.g., [BJO04, BJKR00, Bra72, GPS99, HPS92]; and random walks, see e.g., [GP14].

Proof of Lemma 7.8 (sketch). Let $(p_-(n))$ and $(p_+(n))$ be the transition matrices

$$\begin{aligned} (p_-(n))_{xy} &: x \in V_n, y \in V_{n-1}, \text{ transition from vertex on } V_n \text{ to } V_{n-1} \\ (p_+(n))_{yz} &: y \in V_n, z \in V_{n+1}, \text{ transition from vertex on } V_n \text{ to } V_{n+1}, \text{ see Fig 7.8,} \end{aligned}$$

with row/column index picked from vertices in the respective levels.

The product of $C^{-1}E$ in (7.16) is then (see Fig 7.9)

$$(C^{-1}E)_{xy}^m = \text{Prob}(\text{transition from vertex } x \text{ to vertex } y \text{ in time } m). \quad (7.20)$$

□

Remark 7.9. Under the assumption in Theorem 5.13 and Theorem 6.2 one may show that in fact B (see (6.4)) is *Martin-boundary* (see [Saw97, DJS12]) for the random walk on V defined by

$$p_{xy} := \frac{c_{xy}}{c(x)} \quad \text{for } (x, y) \in E. \quad (7.21)$$

Proof. (sketch) Let G_P be the random-walk Green's function from (7.16) and Lemma 7.8. Set

$$K_{\text{Martin}}(x, y) := \frac{G_P(x, y)}{G_P(o, y)}. \quad (7.22)$$

Then the argument from Theorem 5.13 shows that $K_{\text{Martin}}(x, \cdot)$ extends to B , and that

$$h(x) = \int_B \tilde{h}(b) K_{\text{Martin}}(x, b) d\mu^{(\text{Markov})}(b) \quad (7.23)$$

holds for all $h \in \text{Harm} = \mathcal{H}_E \cap \{h : \Delta h = 0\} = \mathcal{H}_E \cap \{h : \mathbb{P}h = h\}$. □

Example 7.10. For the transition matrix $C^{-1}E = P$, computed with the system in Fig 7.5 of transition probabilities, we get the following:

$$\begin{aligned} p_{i,i} &= 0 \\ p_{i,i+1} &= p_+(i), \text{ and} \\ p_{i,i-1} &= p_-(i), \quad \forall i \in \mathbb{Z}, \end{aligned} \quad (7.24)$$

with the remaining matrix-entries zero. For the computation of the matrix powers P^m , $m = 1, 2, \dots$, we make the following simplification: $p_+(i) = p_+$, and $p_-(i) = p_-$.

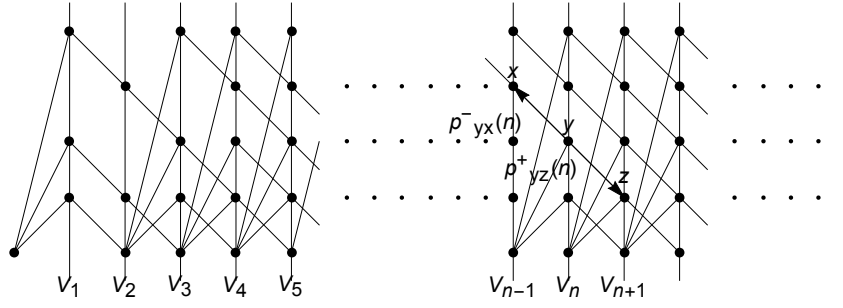
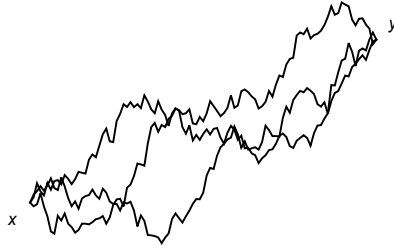


FIGURE 7.8.

FIGURE 7.9. Current flows from vertex x to vertex y .

This then reduces to the following binomial model:

$$P = \begin{pmatrix} \text{diagonal} & & & & \\ & \ddots & & & \\ & & p_- & 0 & p_+ & 0 \\ & & & p_- & 0 & p_+ \\ & & & & p_- & 0 & p_+ \\ 0 & & & & & \ddots & \ddots & \ddots \\ & & & & & & \text{diagonal} \end{pmatrix}, \quad (7.25)$$

$$P^2 = \begin{pmatrix} \ddots & & & & & & 0 \\ & \text{diagonal} & & & & & \\ & \ddots & 0 & 2p_+p_- & 0 & p_+^2 & \\ & & p_-^2 & 0 & 2p_+p_- & 0 & p_+^2 \\ & & & p_-^2 & 0 & 2p_+p_- & 0 \\ & & & & \ddots & & \ddots \\ 0 & & & & & \ddots & \ddots \\ & & & & & & \text{diagonal} \end{pmatrix}, \quad (7.26)$$

and

$$P^3 = \begin{pmatrix} \ddots & \ddots & & & & & & 0 \\ & \ddots & 0 & 3p_+p_-^2 & 0 & 3p_+^2p_- & 0 & p_+^3 \\ & p_-^3 & 0 & 3p_+p_-^2 & 0 & 3p_+^2p_- & 0 & p_+^3 \\ & & p_-^3 & 0 & 3p_+p_-^2 & 0 & 3p_+^2p_- & 0 \\ 0 & & & \ddots & \ddots & \ddots & \text{diagonal} & \ddots \\ & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (7.27)$$

Below we include a sample of matrix-entries for this binomial model:

- Even powers of the transition-matrix P

$$P_{i,i+2k}^{2m} = \binom{2m}{m-k} p_+^{m+k} p_-^{m-k}, \text{ and}$$

$$P_{i,i-2k}^{2m} = \binom{2m}{m-k} p_+^{m-k} p_-^{m+k};$$

where $k = 0, 1, \dots, m$.

- Odd powers of the transition-matrix P

$$P_{i,i+1+2k}^{2m+1} = \binom{2m+1}{m-k} p_+^{m+k+1} p_-^{m-k}, \text{ and}$$

$$P_{i,i-1-2k}^{2m+1} = \binom{2m+1}{m-k} p_+^{m-k} p_-^{m+k+1}.$$

So for the $\infty \times \infty$ matrix G_P in (7.16) we get:

$$\begin{aligned} (G_P)_{i,i+2k} &= \sum_{m=0}^{\infty} \binom{2m}{m-k} p_+^{m+k} p_-^{m-k}; \text{ and} \\ (G_P)_{i,i+2k+1} &= \sum_{m=0}^{\infty} \binom{2m+1}{m-k} p_+^{m+k+1} p_-^{m-k}. \end{aligned}$$

As a result, (7.16) yields an explicit formula for $K_{i,j} = \langle v_i, v_j \rangle_{\mathcal{H}_E}$; see (7.16) and (7.14).

Theorem 7.11. *The Δ -Green's function K in (7.28) has an explicit (and closed form) expression; for example, its diagonal entries are:*

$$K_{i,i} = \frac{1}{c(i) \sqrt{1 - 4p_+ (1 - p_+)}} \quad \text{when } p_+ \neq \frac{1}{2}.$$

Proof. The infinite sums used in computation of $(G_P)_{i,j}$; and therefore of

$$K_{i,j} = (G_P)_{i,j} / c(j) \tag{7.28}$$

can be computed with the use of *generating functions* for the associated binomial coefficients. For example,

$$\sum_{n=0}^{\infty} \lambda^n \binom{2m}{m} = \frac{1}{\sqrt{1 - 4\lambda}}, \text{ setting } \lambda := p_+ p_-; \tag{7.29}$$

and so we get

$$(G_P)_{i,i} = \frac{1}{\sqrt{1 - 4p_+ p_-}}; \tag{7.30}$$

and therefore

$$K_{i,i} = \frac{1}{c(i) \sqrt{1 - 4p_+ (1 - p_+)}} = \langle v_i, v_i \rangle_{\mathcal{H}_E} = d_{res}(o, i), \tag{7.31}$$

which is the desired conclusion. \square

Note that to get absolute convergence in these series the requirement on p_+ is that $p_+ \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. (In this case, the resistance metric is bounded. We have $\sum_j \frac{1}{c(j)} < \infty$.) The degenerate case is $p_+ = p_- = \frac{1}{2}$. However the latter degenerate case can easily be computed by hand. It is the case of constant conductance function, $c_{i,i+1} = 1$.

For more details on this and related binomial models, see [AJ15, AJSV13, BV14].

Remark 7.12 (On general Bratteli diagrams). While the formulas (7.20)-(7.31) are derived subject to rather restricting assumptions, an inspection of the arguments shows

that the ideas work for general Bratteli-diagrams; but then with modifications; see below:

Given a Bratteli diagram with vertex-set $V = \{0\} \cup \bigcup_{n=1}^{\infty} V_n$, and vertices V_n corresponding to levels $n = 1, 2, \dots$ (see Fig 7.8), we then have the following transition matrices:

$$\begin{cases} p^+(n)_{x,y} & x \in V_n, y \in V_{n+1}, \text{ and} \\ p^-(n)_{x,z} & x \in V_n, z \in V_{n-1}. \end{cases} \quad (7.32)$$

Therefore, in computing transition-probabilities,

$$\text{Prob}(x \longrightarrow y \text{ in } 2m \text{ iterations}), \quad (7.33)$$

we specialize to $x \in V_n$, and $y \in V_{n+2k}$. Rather than the easy formulas $\binom{2m}{m+k} p_+^{m+k} p_-^{m-k}$ from the proof in Example 7.10, we now instead get a sum of products of non-commutative matrices:

$$P_{w_1} P_{w_2} \cdots P_{w_{2m}} \quad (7.34)$$

where $w = (w_1, w_2, \dots, w_{2m})$ is a finite word in the two-letter alphabet $\{\pm\}$, i.e., $w_i \in \{\pm\}$; but the estimates from before carry over; and we again arrive at an expression for the Green's function $(G_P)_{x,y}$, $x, y \in V$, analogous to (7.20)-(7.31).

Example 7.13 (The N -ary tree). Fix $N > 1$. Let $b \in \mathbb{R}_+$, $b > 1$, be fixed, and set

$$c(n) := b^n, \quad x \in V_n, y \in V_{n+1}; \quad (7.35)$$

then (see 7.32), we have (see Fig 7.10):

$$\begin{cases} p^+(n)_{xy} &= \frac{b}{1 + Nb}, \\ p^-(n)_{xz} &= \frac{1}{1 + Nb}, \text{ and} \\ c(n)_x &= b^{n-1} (1 + Nb) \end{cases} \quad (7.36)$$

where $x \in V_n, y \in V_{n+1}, z \in V_{n-1}$.

Generalizing (7.30), we get

$$(G_P)_{x,x'} = \frac{Nb + 1}{Nb - 1} \quad (7.37)$$

for all $x, x' \in V_n$; and

$$d_{res}(\emptyset, x) = \frac{1}{(1 + Nb) b^{n-1}}; \quad (7.38)$$

and $d_{res}(x, B) < \infty$.

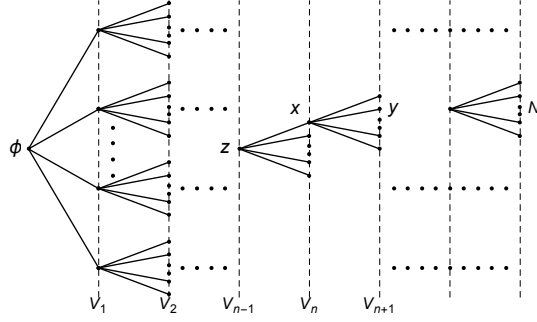


FIGURE 7.10. N -ary tree; the vertices at level n are denoted V_n , $n = 0, 1, \dots$, $V_0 = \{\emptyset\}$, the empty word.

One can show that, if $\#V_1 < \#V_2 < \dots$ (strictly increasing), then

$$\dim \{f : \Delta f = 0\} = \infty. \quad (7.39)$$

8. THE PATH-SPACE MARKOV MEASURE VS THE POISSON-MEASURE ON B

Here, we consider a class of models (V, E, c) :

- (i) V = vertices;
- (ii) $E \subset V \times V \setminus (\text{diagonal})$ edges;
- (iii) $c : E \rightarrow \mathbb{R}_+$ conduction function, which induces a resistance metric d_{res} ;
- (iv) Δ graph Laplacian;
- (v) \mathcal{H}_E the energy-Hilbert space;
- (vi) $B = M \setminus V$, where M is the metric completion;
- (vii) path space $\Omega = \{\omega = (\omega_i) \mid \omega_i \in V, (\omega_i \omega_{i+1}) \in E, \forall i \in \mathbb{N}\}$;
- (viii) Set $\pi_i(\omega) = \omega_i \in V$ (vertex at time i), $i = 0, 1, 2, \dots$, and

$$\Omega_x = \{\omega \in \Omega \mid \pi_0(\omega) = x\};$$

- (ix) Set $p_{xy} = c_{xy}/c(x)$, $(xy) \in E$;
- (x) $\mu_x^{(M)}$: Markov measure on Ω_x , $x \in V$ with transition

$$\mu_x^{(M)}(\text{cylinder}) = p_{x\omega_1} p_{\omega_1\omega_2} \cdots \text{ (see (ix))}. \quad (8.1)$$

In more detail, a cylinder set $\subset \Omega$ is specified by a finite word $(xx_1x_2 \cdots x_n)$ of vertices such that $(xx_1), (x_1x_2), \dots$ are edges (i.e., in E). Then set

$$C_{xx_1 \cdots x_n} = \{\omega \in \Omega \mid \pi_0(\omega) = x, \pi_i(\omega) = x_i, 1 \leq i \leq n\}. \quad (8.2)$$

Formula (8.1) then reads as follows:

$$\mu_x^{(M)}(C_{xx_1x_2\cdots x_n}) = p_{xx_1}p_{x_1x_2}\cdots p_{x_{n-1}x_n} \quad (8.3)$$

The following is known, see e.g., [Doo72, DJ07]:

Lemma 8.1. *There is a 1-1 correspondence between harmonic functions h on V , on the one hand, and shift-invariant L^1 -functions F on Ω , on the other. It is given as follows:*

Let \mathbb{E} denote the expectation computed w.r.t. the Markov-measure on Ω , then

$$h(x) = \mathbb{E}(F | \pi_0 = x), \quad x \in V, \quad (8.4)$$

is harmonic of finite energy iff there is a shift-invariant L^1 -function F on Ω such that (8.4) holds. (In (8.4), the symbol $\mathbb{E}(\cdot | \pi_0 = x)$ refers to conditional expectation.)

Proof. (see [DJ07]) use the formula

$$(\Delta h)(x) = c(x)(h(x) - (\mathbb{P}h)(x)), \quad x \in V, \quad (8.5)$$

where

$$(\mathbb{P}f)(x) = \sum_{y \sim x} p_{xy} f(y), \quad (8.6)$$

and $p_{xy} = c_{xy}/c(x)$ for $(x, y) \in E$. □

Definition 8.2. *Class $A(V, E, c, d_{res})$:*

$$\lim_{k, l \rightarrow \infty} d_{res}(\pi_k(\omega), \pi_l(\omega)) = 0 \quad (8.7)$$

for all $\omega \in \Omega$, or in a “big” subset of Ω .

Remark 8.3. A large subset of Bratteli diagram will be of *class A*, i.e., that (8.7) holds; for example, if

$$\sum_n r(n) < \infty \quad (8.8)$$

where $r(n)$ denotes the resistance $V_n \rightarrow V_{n+1}$ between vertices of level n and level $n+1$. So (8.8) \implies (8.7); but (8.7) holds much more generally.

Proposition 8.4. *Assume (8.7). Then there is a well defined mapping: $\Omega \xrightarrow{\Phi} B$, given by*

$$\Omega \longrightarrow (\text{Cauchy-sequences}) \longrightarrow (\text{Cauchy-sequences}) / \sim \quad (8.9)$$

$$\omega \longmapsto \Psi(\omega) = \text{class}(\pi_0(\omega), \pi_1(\omega), \pi_2(\omega), \dots) \quad (8.10)$$

where \sim on Cauchy-sequences

$$(\tilde{x}) \sim (\tilde{y}) \iff \lim_{\substack{\text{Def} \\ i \rightarrow \infty}} d_{res}(x_i, y_i) = 0. \quad (8.11)$$

Theorem 8.5. Let $V, E, c, p_{xy} = c_{xy}/c(x)$, $\mu_x^{(M)}$ Markov measure, and let $\Psi : \Omega \rightarrow B$ be the mapping in (8.10) of Proposition 8.4. Then

$$\left\{ \mu_x^{(M)} \circ \Psi^{-1} \right\}_{x \in V} \quad (8.12)$$

constitutes the Poisson-measure on B in Theorem 6.2; i.e., if $S \in \mathcal{B}(B)$, $S \subset B$ is a given Borel subset, then the measure in (8.12) is $\mu_x^{(M)}(\Psi^{-1}(S))$, where

$$\Psi^{-1}(S) = \{\omega \in \Omega \mid \Psi(\omega) \in S\}.$$

Proof. (sketch) Set $\mu_x := \mu_x^{(M)} \circ \Psi^{-1}$, we then need to prove that

$$h(x) = \int_B \tilde{h} d\mu_x \quad (8.13)$$

holds for all harmonic function $h \in \mathcal{H}_E$, i.e., $\|h\|_{\mathcal{H}_E} < \infty$, $\Delta h = 0$ ($\iff \mathbb{P}h = h$), and where $\tilde{h} \in C(B)$ is the restriction to B of the extension from

$$\begin{array}{ccccc} V & \longrightarrow & M & \longrightarrow & B \\ & & \tilde{h} & & \tilde{h}|_B \end{array}$$

With this, we can check directly that μ_x satisfies (8.13), and so μ_x must be the Poisson-measure by uniqueness. \square

9. BOUNDARY AND INTERPOLATION

Theorem 9.1. Let $V, E, c, \Delta, d_{res}, \mathcal{H}_E$, and B be as above (see (i)-(x) in Section 8). We pick a base-point $o \in V$, and dipoles $v_x = v_{(xo)}$ s.t. $v_x(o) = 0$, and we set

$$K(x, y) = \langle v_x, v_y \rangle_{\mathcal{H}_E} = v_x(y) = v_y(x), \quad (9.1)$$

the Green's function for Δ . Finally, set $Q := Q_{Harm}$ denote the projection of \mathcal{H}_E onto the subspace $Harm = \{h \in \mathcal{H}_E \mid \Delta h = 0\}$. For $x \in V$, let μ_x denote the Poisson-measure.

Then we have the following interpolation/boundary formula:

$$f(x) = \sum_{y \in V \setminus \{o\}} K(x, y) (\Delta f)(y) + \int_B \widetilde{(Qf)}(b) d\mu_x(b) \quad (9.2)$$

valid for all $f \in \mathcal{H}_E$, and all $x \in V$.

Proof. From [AJLM13, Jor11], we have that the projection $Q^\perp = I_{\mathcal{H}_E} - Q$ is given by

$$(Q^\perp f) = \sum_{y \in V} (\Delta f)(y) v_y = \sum_{y \in V} \underbrace{|v_y\rangle\langle\delta_y|}_{\text{Dirac-notation}} (f); \quad (9.3)$$

or equivalently,

$$(Q^\perp f)(x) = \sum_{y \in V \setminus \{o\}} K(x, y) (\Delta f)(y), \quad \forall x \in V. \quad (9.4)$$

Since $f = (Q^\perp f) + (Qf)$ with $Qf \in \text{Harm}(\subset \mathcal{H}_E)$, the desired formula (9.2) follows from the Poisson-representation:

$$(Qf)(x) = \int_B \widetilde{(Qf)}(b) d\mu_x(b).$$

We have used the following: □

Lemma 9.2. *The operator $A = Q^\perp$ in (9.3) indeed is a projection in \mathcal{H}_E , i.e., $A^2 = A = A^*$ where the adjoint $*$ is computed w.r.t. the \mathcal{H}_E -inner product.*

Proof. We have $A = \sum_x |v_x\rangle\langle\delta_x|$, and so

$$\begin{aligned} A^2 &= \sum_x \sum_y (|v_x\rangle\langle\delta_x|) (|v_y\rangle\langle\delta_y|) \\ &= \sum_x \sum_y \langle\delta_x, v_y\rangle_{\mathcal{H}_E} |v_x\rangle\langle\delta_y| \\ &= \sum_x \sum_y \delta_{xy} |v_x\rangle\langle\delta_y| = \sum_x |v_x\rangle\langle\delta_x| = A. \end{aligned}$$

But we also have for $f, g \in \mathcal{H}_E$, that

$$\begin{aligned} \langle f, Ag \rangle_{\mathcal{H}_E} &= \sum_x \overline{f(x)} (\Delta g)(x) \\ &= \sum_x \overline{(\Delta f)(x)} g(x) = \langle Af, g \rangle_{\mathcal{H}_E}, \end{aligned}$$

where we use Lemma 2.7 (1), so $A = A^*$.

From this, we get operator-norm $\|A\|_{\mathcal{H}_E \rightarrow \mathcal{H}_E} = 1$. It is immediate from (9.3) that $Ah = 0$ for all $h \in \text{Harm}$, and further that $A = \text{proj onto } \mathcal{H}_E \ominus \text{Harm}$. Recall $\mathcal{H}_E \ominus \text{Harm} = \mathcal{H}_E$ -norm closure of $\{\delta_x \mid x \in V\}$. □

Remark 9.3. Note that the function $K(\cdot, \cdot)$ from (9.1)-(9.2) is a Green's function of the Laplacian Δ . Recall Δ from Lemma 2.7 has the following $\infty \times \infty$ matrix-representation:

$$\Delta_{xy} = \begin{cases} c(x) & \text{if } x = y \\ -c_{xy} & \text{if } x \neq y \text{ but } (xy) \in E \\ 0 & \text{otherwise;} \end{cases} \quad (9.5)$$

i.e., as an $\infty \times \infty$ matrix Δ has the following banded form

$$\Delta = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \ddots & -c_{xz} & & \mathbf{0} \\ & -c_{xz} & c(x) & -c_{xy} & \\ \mathbf{0} & & -c_{xy} & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix} = C - E \quad (9.6)$$

where $C = \text{diag}\{c(x) \mid x \in V\}$

$$C = \begin{pmatrix} c(x) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & c(x') & \\ & & & \ddots \end{pmatrix} \quad (9.7)$$

and

$$E = \begin{pmatrix} 0 & \ddots & & \\ \ddots & \ddots & c_{xy} & \\ & c_{xy} & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}; \quad (9.8)$$

both $\infty \times \infty$ matrices, C with the sequence $(c(x))_{x \in V}$ as diagonal entries, and E with 0 in the diagonal but with (c_{xy}) as entries in the off diagonal entries.

One checks from Lemma 2.7, that the Green's inversion then holds:

$$\sum_{z \in V'} \Delta_{xz} K(z, y) = \delta_{x,y}, \quad \forall (x, y) \in V' \times V' \quad (9.9)$$

where $K(\cdot, \cdot)$ in (9.9) is the $\infty \times \infty$ matrix introduced in (9.1). So infimum about the resistance metric results from an inversion of the matrix (Δ_{xy}) in (9.6) above.

Corollary 9.4. *For every $f \in \mathcal{H}_E$ with $f(o) = 0$, we have the following representation:*

$$\|f\|_{\mathcal{H}_E}^2 = \langle f, \Delta f \rangle_{l^2} + \int_{B_{Markov}} |\widetilde{Q}f|^2 d\mu^{(Markov)} \quad (9.10)$$

where

$$\langle f, \Delta f \rangle_{l^2} = \sum_{x \in V} \overline{f(x)} (\Delta f)(x) \quad (9.11)$$

and where $\mu^{(Markov)}$ is the Markov measure from Theorem 8.5.

Proof. First, by Theorem 9.1 we have $f = Q^\perp f + Qf$ as an orthogonal splitting, relative to the \mathcal{H}_E -inner product. Hence

$$\|f\|_{\mathcal{H}_E}^2 = \|Q^\perp f\|_{\mathcal{H}_E}^2 + \|Qf\|_{\mathcal{H}_E}^2. \quad (9.12)$$

For the first term in (9.12), we have

$$\begin{aligned} \|Q^\perp f\|_{\mathcal{H}_E}^2 &= \langle f, Q^\perp f \rangle_{\mathcal{H}_E} \\ &\stackrel{\text{(by (9.3))}}{=} \sum_x (\Delta f)(x) \langle f, v_x \rangle_{\mathcal{H}_E} \\ &= \sum_x \overline{f(x)} (\Delta f)(x) = \langle f, \Delta f \rangle_{l^2}. \end{aligned}$$

For the second term in (9.12), we get, using Proposition 8.4 and Theorem 8.5,

$$\|Qf\|_{\mathcal{H}_E}^2 = \int_{B_{Markov}} |\widetilde{Q}f|^2 d\mu^{(Markov)}; \quad (9.13)$$

see also [Anc90]. The desired conclusion (9.10) now follows. \square

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